A closed-form solution for optimal mean-reverting trading strategies

Alexander Lipton* and Marcos Lopez de Prado†

Abstract

When prices reflect all available information, they oscillate around an equilibrium level. This oscillation is the result of the temporary market impact caused by waves of buyers and sellers. This price behavior can be approximated through an Ornstein-Uhlenbeck (OU) process.

Market makers provide liquidity in an attempt to monetize this oscillation. They enter a long position when a security is priced below its estimated equilibrium level, and they enter a short position when a security is priced above its estimated equilibrium level. They hold that position until one of three outcomes occur: (1) they achieve the targeted profit; (2) they experience a maximum tolerated loss; (3) the position is held beyond a maximum tolerated horizon.

All market makers are confronted with the problem of defining profit-taking and stop-out levels. More generally, all execution traders holding a particular position for a client must determine at what levels an order must be fulfilled. Those optimal levels can be determined by maximizing the trader's Sharpe ratio in the context of OU processes via Monte Carlo experiments, [12]. This paper develops an analytical framework and derives those optimal levels by using the method of heat potentials, [6, 8].

1 Introduction

Mean-reverting trading strategies in various contexts have been studied for decades. For instance, Elliott *et al.* explained how mean-reverting processes might be used in pairs trading, and developed several method for parameter estimation, [3]; Avellaneda and Lee used mean-reverting processes for pairs trading, and modeled the hitting time to find the exit rule of the trade, [1]; Bertram developed some analytic formulae for statistical arbitrage trading where the security price follows an Ornstein–Uhlenbeck (O-U) process, [2]; Lopez de Prado (Chapter 13) considered trading rules for discrete time mean-reverting trading strategies, and found optimal trading rules using Monte Carlo simulations, [12].

When prices reflect all available information, they oscillate around an equilibrium level. This oscillation is the result of the temporary market impact caused by waves of buyers and sellers. The resulting price behavior can be approximated through an O-U process. The parameters of the process might be estimated using historical data.

Market makers provide liquidity in an attempt to monetize this oscillation. They enter a long position when a security is priced below its estimated equilibrium level, and they enter a short position when a security is priced above its estimated equilibrium level. They hold that position until one of three outcomes occur: (A) they achieve the targeted profit; (B) they experience a maximum tolerated loss; (C) the position is held beyond a maximum tolerated horizon.

All traders are confronted with the problem of defining profit-taking and stop-out levels. More generally, all execution traders holding a particular position for a client must determine at what levels an order must be fulfilled. Lopez de Prado (Chapter 13) explains how to determine those optimal levels in the sense of maximizing the trader's Sharpe ratio (SR) in the context of O-U processes via Monte Carlo experiments, [12]. Although Lopez de Prado (p. 192) conjectured the existence of an analytical solution to this problem,

*The Jerusalem School of Business Administration, The Hebrew University of Jerusalem, Jerusalem, Israel; Connection Science and Engineering, Massachusetts Institute of Technology, Cambridge, MA, USA; Investimizer, Chicago, IL, USA; SilaMoney, Portland, OR, USA. E-mail: alexlipt@mit.edu

†Operations Research and Information Engineering, Cornell University, New York, NY, USA; Investimizer, Chicago, IL, USA; True Positive Technologies, New York, NY, USA. E-mail ml863@cornell.edu

he identified it as an open problem. In this paper we solve the important problem of finding optimal trading rules analytically by using the method of heat potentials. These optimal profit-taking and stop-loss trading rules for mean-reverting trading strategies provide the algorithm that must be followed to exit a position. To put it differently, we find the optimal exit corridor to maximize the SR of the strategy.

The method of heat potential is a highly powerful and versatile approach popular in mathematical physics, see, e.g. [15, 14, 4, 16] among others. It has been successfully used in numerous important fields such as thermal engineering, nuclear engineering, and material science. However, it is not particularly popular in mathematical finance, in spite of the fact that the first important use case was given by Lipton almost twenty years ago. Specifically, Lipton considered pricing barrier options with curvilinear barriers, see [5], Section 12.2.3, pp. 462–467. More recently, Lipton and Kaushansky described several important financial applications of the method, see [6, 7, 8, 9].

The SR is defined as the ratio between the expected returns of an execution algorithm and the standard deviation of the same returns. The returns are computed as the logarithmic ratio between the exit and entry prices, times the sign of the order side (+1 for a sell order, −1 for a buy order). Our choice of the SR as an objective function is due to two reasons: (A) The SR is the most popular criterion for investment efficiency, [10]; (B) The SR can be understood as a t-value of the estimated gains, and modelled accordingly for inferential purposes. The distributional properties of the SR are well-known, and this statistic can be deflated when the assumption of normality is violated, [11].

Having an analytical estimation of the optimal profit-taking and stop-out levels allows traders to deploy tactical execution algorithms, with maximal expected SR. Rather than deriving an "all-weather" execution algorithm, which supposedly works under every market regime, traders can use our analytical solution for deploying the algorithm that maximizes the SR under the prevailing market regime, [13].

2 Definitions of variables

Suppose an investment strategy S invests in $i = 1, ...I$ opportunities or bets. At each opportunity i, S takes a position of m_i units of security X, where $m_i \in (-\infty, \infty)$. The transaction that entered such opportunity was priced at a value $m_i P_{i,0}$, where $P_{i,0}$ is the average price per unit at which the m_i securities were transacted. As other market participants transact security X, we can mark-to-market (MtM) the value of that opportunity i after t observed transactions as $m_i P_{i,t}$. This represents the value of opportunity i if it were liquidated at the price observed in the market after t transactions. Accordingly, we can compute the MtM profit/loss of opportunity i after t transactions as $\pi_{i,t} = m_i(P_{i,t} - P_{i,0})$.

A standard trading rule provides the logic for exiting opportunity i at $t = T_i$. This occurs as soon as one of two conditions is verified:

- $\pi_{i,T_i} \geq \overline{\pi}$, where $\overline{\pi} > 0$ is the profit-taking threshold.
- $\pi_{i,T_i} \leq \underline{\pi}$, where $\underline{\pi} < 0$ is the stop-loss threshold.

Because $\underline{\pi} < \overline{\pi}$, one and only one of the two exit conditions can trigger the exit from opportunity i. Assuming that opportunity i can be exited at T_i, its final profit/loss is π_{i,T_i}. At the onset of each opportunity, the goal is to realize an expected profit

$$E_0[\pi_{i,T_i}] = m_i(E_0[P_{i,T_i}] - P_{i,0}),$$

where $E_0[P_{i,T_i}]$ is the forecasted price and $P_{i,0}$ is the entry level of opportunity i.

3 Parameter estimation

Consider the discrete O-U process on a price series $\{P_{i,t}\}$:

$$P_{i,t} - E_0[P_{i,T_i}] = \kappa \left(E_0[P_{i,T_i}] - P_{i,t-1}\right) + \sigma\varepsilon_{i,t},$$

such that the random shocks are IID distributed $\varepsilon_{i,t} \sim \mathcal{N}(0, 1)$. The seed value for this process is $P_{i,0}$, the level targeted by opportunity i is $E_0[P_{i,T_i}]$, and κ determines the speed at which $P_{i,0}$ converges towards $E_0[P_{i,T_i}]$.

We estimate the input parameters $\{\sigma, \kappa\}$, by linearizing the above equation as:

$$P_{i,t} = E_0[P_{i,T_i}] + \kappa \left(E_0[P_{i,T_i}] - P_{i,t-1}\right) + \vartheta_t.$$

We can then form vectors X and Y by stacking the opportunities:

$$X = \begin{bmatrix} E_0[P_{0,T_0}] - P_{0,0} \\ E_0[P_{0,T_0}] - P_{0,1} \\ \cdots \\ E_0[P_{0,T_0}] - P_{0,T-1} \\ \cdots \\ E_0[P_{I,T_I}] - P_{I,0} \\ \cdots \\ E_0[P_{I,T_I}] - P_{I,T-1} \end{bmatrix}; \quad Y = \begin{bmatrix} P_{0,1} \\ P_{0,2} \\ \cdots \\ P_{0,T} \\ \cdots \\ P_{I,1} \\ \cdots \\ P_{I,T} \end{bmatrix}; \quad Z = \begin{bmatrix} E_0[P_{0,T_0}] \\ E_0[P_{0,T_0}] \\ \cdots \\ E_0[P_{0,T_0}] \\ \cdots \\ E_0[P_{I,T_I}] \\ \cdots \\ E_0[P_{I,T_I}] \end{bmatrix}.$$

Applying OLS on the above equation, we can estimate the original O-U parameters as follows:

$$\begin{aligned} \hat{\kappa} &= \frac{cov\,[Y, X]}{cov\,[X, X]}, \\ \hat{\vartheta}_t &= Y - Z - \hat{\kappa} X, \\ \hat{\sigma} &= \sqrt{cov\left[\hat{\vartheta}_t, \hat{\vartheta}_t\right]}, \end{aligned}$$

where, as usual, $cov\,[.,.]$ is the covariance operator.

We use the above estimations to find optimal stop-loss and take-profit estimates.

4 Explicit problem formulation

In this rather technical section, we perform transformations in order to formulate the problem in terms of heat potentials.

Consider a long investment strategy S and suppose profit/loss opportunity is driven by an O-U process (see [12] among many others):

$$dx' = \kappa' \left(\theta' - x'\right) dt' + \sigma' dW_{t'}, \quad x'(0) = 0, \tag{1}$$

and a trading rule $R = \{\underline{\pi}', \overline{\pi}', T'\}$, $\underline{\pi}' < 0$, $\overline{\pi}' > 0$. According to the trading rule, we exit the trade either when: (A) the price hits $\overline{\pi}'$ to take profit; (B) the price hits $\underline{\pi}'$ to stop losses; (C) the trade expires at $t' = T'$. For a short investment strategy, the roles of $\{\underline{\pi}', \overline{\pi}'\}$ are reversed - profits equal to $-\underline{\pi}'$ are taken when when the price hits $\underline{\pi}'$, and losses equal $-\overline{\pi}'$ are realized when the price hits $\overline{\pi}'$. Given the fact that the reflection $x' \to -x'$ leaves the initial condition unchanged and transforms the original O-U process into the O-U process of the form

$$dx' = \kappa' \left(-\theta' - x'\right) dt' + \sigma' dW_{t'}, \quad x'(0) = 0,$$

we can restrict ourselves to the case $\theta' \geq 0$. More explicitly and intuitively, we go long when $\theta' \geq 0$ and short when $\theta' < 0$. Assuming that we know the trading rule $\{\underline{\pi}'\left(\theta', T'\right), \overline{\pi}'\left(\theta', T'\right), T'\}$ for $\theta' \geq 0$, the corresponding trading rule for $\theta' < 0$ has the form

$$\{\underline{\pi}'\left(\theta', T'\right), \overline{\pi}'\left(\theta', T'\right), T'\} = \{-\underline{\pi}'\left(-\theta', T'\right), -\overline{\pi}'\left(-\theta', T'\right), T'\}.$$

Thus, we are interested in the maximization of the SR for nonnegative $\theta' \geq 0$. We formulate this mathematically below.

For a given T', we define the stopping time $\iota' = \inf\{t' : x_{t'} = \overline{\pi}' \text{ or } x_{t'} = \underline{\pi}' \text{ or } t' = \underline{T}'\}$. We wish to determine optimal $\overline{\pi}' > 0, \underline{\pi}' < 0$,to maximize the SR,

$$\mathrm{SR} = \frac{\mathbb{E}\left\{x'_\iota/\iota'\right\}}{\sqrt{\mathbb{E}\left\{x'^2_{\iota'}/\iota'^2\right\} - \left(\mathbb{E}\left\{x'_{\iota'}/\iota'\right\}\right)^2}},$$

We also need to know the expected duration of the trade,

$$\mathrm{DUR} = \mathbb{E}\left\{\iota'\right\}.$$

It is important to understand what are the natural units associated with the O-U process (1). To this end we can use its steady-state. The steady-state expectation of the above process is θ', while its standard deviation is given by $\Omega' = \sigma'/\sqrt{2\kappa'}$.

In order to calculate the corresponding SR and DUR we proceed as follows. We solve three terminal boundary value problems (TBVPs) of the form

$$\begin{gathered} E'_{t'}\left(t',x'\right) + \kappa'\left(\theta' - x'\right)E'_{x'}\left(t',x'\right) + \tfrac{1}{2}\sigma^2 E'_{x'x'}\left(t',x'\right) = 0, \\ E'\left(t',\overline{\pi}\right) = \tfrac{\overline{\pi}}{t'}, \quad E'\left(t',\underline{\pi}\right) = \tfrac{\underline{\pi}}{t'}, \\ E'\left(T',x'\right) = \tfrac{x'}{T'}, \end{gathered}$$

$$\begin{gathered} F'_{t'}\left(t',x'\right) + \kappa'\left(\theta' - x'\right)F'_{x'}\left(t',x'\right) + \tfrac{1}{2}\sigma'^2 F'_{x'x'}\left(t',x'\right) = 0, \\ F'\left(t',\overline{\pi}'\right) = \tfrac{\overline{\pi}'^2}{t'^2}, \quad F'\left(t',\underline{\pi}'\right) = \tfrac{\underline{\pi}'^2}{t'^2}, \\ F'\left(T',x'\right) = \tfrac{x'^2}{T'^2}, \end{gathered}$$

and

$$\begin{gathered} G'_{t'}\left(t',x'\right) + \kappa'\left(\theta' - x'\right)G'_{x'}\left(t',x'\right) + \tfrac{1}{2}\sigma^2 G'_{x'x'}\left(t',x'\right) = 0, \\ G'\left(t',\overline{\pi}\right) = t', \quad G'\left(t',\underline{\pi}\right) = t', \\ G'\left(T',x'\right) = T'. \end{gathered}$$

We represent the SR and DUR as

$$\mathrm{SR} = \frac{E'\left(0,0\right)}{\sqrt{F'\left(0,0\right) - \left(E'\left(0,0\right)\right)^2}},$$

$$\mathrm{DUR} = G'\left(0,0\right).$$

As usual, an appropriate scaling is helpful to remove superfluous parameters. To this end, we define

$$t = \kappa' t', \quad T = \kappa' T', \quad x = \frac{\sqrt{\kappa'}}{\sigma'}x', \quad \overline{\pi} = \frac{\sqrt{\kappa'}}{\sigma'}\overline{\pi}', \quad \underline{\pi} = \frac{\sqrt{\kappa'}}{\sigma'}\underline{\pi}', \quad \theta = \frac{\sqrt{\kappa'}}{\sigma'}\theta', \quad E = \frac{E'}{\sqrt{\kappa'}\sigma'}, \quad F = \frac{F'}{\kappa'\sigma'^2},$$

and get

$$dx = \left(\theta - x\right)dt + dW_t,$$

in the domain

$$\underline{\pi} \le x \le \overline{\pi}, \quad 0 \le t \le T.$$

The steady-state distribution has the expectation of θ, and the standard deviation $\Omega = 1/\sqrt{2}$.

We write the problems of interest as follows:

$$\begin{gathered} E_t\left(t,x\right) + \left(\theta - x\right)E_x\left(t,x\right) + \tfrac{1}{2}E_{xx}\left(t,x\right) = 0, \\ E\left(t,\overline{\pi}\right) = \tfrac{\overline{\pi}}{t}, \quad E\left(t,\underline{\pi}\right) = \tfrac{\underline{\pi}}{t}, \\ E\left(T,x\right) = \tfrac{x}{T}, \end{gathered}$$

$$\begin{gathered} F_t\left(t,x\right) + \left(\theta - x\right)F_x\left(t,x\right) + \tfrac{1}{2}F_{xx}\left(t,x\right) = 0, \\ F\left(t,\overline{\pi}\right) = \tfrac{\overline{\pi}^2}{t^2}, \quad F\left(t,\underline{\pi}\right) = \tfrac{\underline{\pi}^2}{t^2}, \\ F\left(T,x\right) = \tfrac{x^2}{T^2}, \end{gathered}$$

$$\begin{gathered} G_t\left(t,x\right) + \left(\theta - x\right)G_x\left(t,x\right) + \tfrac{1}{2}G_{xx}\left(t,x\right) = 0, \\ G\left(t,\overline{\pi}\right) = t, \quad G\left(t,\underline{\pi}\right) = t, \\ G\left(T,x\right) = T, \end{gathered}$$

$$\mathrm{SR} = \frac{E\left(0,0\right)}{\sqrt{F\left(0,0\right) - \left(E\left(0,0\right)\right)^2}}, \tag{2}$$

$$\mathrm{DUR} = G(0,0).$$

We wish to use the method of heat potentials to solve the above TBVPs. First, we define

$$\tau = T - t,$$

and get initial boundary value problems (IBVPs):

$$\begin{array}{c} E_\tau(\tau,x) = (\theta - x) E_x(\tau,x) + \frac{1}{2} E_{xx}(\tau,x), \\ E(\tau,\overline{\pi}) = \frac{\overline{\pi}}{(T-\tau)}, \quad E(\tau,\underline{\pi}) = \frac{\underline{\pi}}{(T-\tau)}, \\ E(0,x) = \frac{x}{T}, \end{array}$$

$$\begin{array}{c} F_\tau(\tau,x) = (\theta - x) F_x(\tau,x) + \frac{1}{2} F_{xx}(\tau,x), \\ F(\tau,\overline{\pi}) = \frac{\overline{\pi}^2}{(T-\tau)^2}, \quad F(\tau,\underline{\pi}) = \frac{\underline{\pi}^2}{(T-\tau)^2}, \\ F(0,x) = \frac{x^2}{T^2}, \end{array}$$

$$\begin{array}{c} G_\tau(\tau,x) = (\theta - x) G_x(\tau,x) + \frac{1}{2} G_{xx}(\tau,x), \\ G(\tau,\overline{\pi}) = (T-\tau), \quad G(\tau,\underline{\pi}) = (T-\tau), \\ G(0,x) = T, \end{array}$$

$$\mathrm{SR} = \frac{E(T,0)}{\sqrt{F(T,0) - (E(T,0))^2}}, \tag{3}$$

$$\mathrm{DUR} = G(T,0).$$

Second, we define

$$\upsilon = \frac{1 - e^{-2\tau}}{2}, \quad \xi = e^{-\tau}(x - \theta),$$

so that

$$\partial_\tau = (1 - 2\upsilon)\partial_\upsilon - \xi\partial_\xi, \quad \partial_x = \sqrt{1 - 2\upsilon}\,\partial_\xi.$$

Accordingly,

$$\begin{array}{c} E_\upsilon(\upsilon,\xi) = \frac{1}{2} E_{\xi\xi}(\upsilon,\xi), \\ E\left(\upsilon, \overline{\Pi(\upsilon)}\right) = \frac{2\overline{\pi}}{\ln\left(\frac{(1-2\upsilon)}{(1-2\Upsilon)}\right)}, \quad E\left(\upsilon, \underline{\Pi(\upsilon)}\right) = \frac{2\underline{\pi}}{\ln\left(\frac{(1-2\upsilon)}{(1-2\Upsilon)}\right)}, \\ E(0,\xi) = -\frac{2(\xi+\theta)}{\ln(1-2\Upsilon)}, \end{array}$$

$$\begin{array}{c} F_\upsilon(\upsilon,\xi) = \frac{1}{2} F_{\xi\xi}(\upsilon,\xi), \\ F\left(\upsilon, \overline{\Pi(\upsilon)}\right) = \frac{4\overline{\pi}^2}{\left(\ln\left(\frac{(1-2\upsilon)}{(1-2\Upsilon)}\right)\right)^2}, \quad F\left(\upsilon, \underline{\Pi(\upsilon)}\right) = \frac{\underline{\pi}^2}{\left(\ln\left(\frac{(1-2\upsilon)}{(1-2\Upsilon)}\right)\right)^2}, \\ F(0,\xi) = \frac{4(\xi+\theta)^2}{(\ln(1-2\Upsilon))^2}, \end{array}$$

$$\begin{array}{c} G_\upsilon(\upsilon,\xi) = \frac{1}{2} G_{\xi\xi}(\upsilon,\xi), \\ G\left(\upsilon, \overline{\Pi(\upsilon)}\right) = \frac{1}{2}\ln\left(\frac{(1-2\upsilon)}{(1-2\Upsilon)}\right), \quad G\left(\upsilon, \underline{\Pi(\upsilon)}\right) = \frac{1}{2}\ln\left(\frac{(1-2\upsilon)}{(1-2\Upsilon)}\right), \\ G(0,\xi) = -\frac{1}{2}\ln(1 - 2\Upsilon), \end{array}$$

$$\mathrm{SR} = \frac{E(\Upsilon,\varpi)}{\sqrt{F(\Upsilon,\varpi) - (E(\Upsilon,\varpi))^2}}, \tag{4}$$

$$\mathrm{DUR} = G(\Upsilon,\varpi).$$

Here

$$\begin{aligned} \Upsilon &= \frac{1 - e^{-2T}}{2}, \quad \varpi = -\sqrt{1 - 2\Upsilon}\,\theta, \\ \overline{\Pi}(\upsilon) &= \sqrt{1 - 2\upsilon}\,(\overline{\pi} - \theta), \quad \underline{\Pi}(\upsilon) = \sqrt{1 - 2\upsilon}\,(\underline{\pi} - \theta). \end{aligned}$$

As usual, we have to account for the initial conditions. To this end, we write

$$E(\upsilon,\xi)=\hat{E}(\upsilon,\xi)-\frac{2(\xi+\theta)}{\ln(1-2\Upsilon)},$$

$$F(\upsilon,\xi)=\hat{F}(\upsilon,\xi)+\frac{4\left(\upsilon+(\xi+\theta)^2\right)}{(\ln(1-2\Upsilon))^2},$$

$$G(\upsilon,\xi)=\hat{G}(\upsilon,\xi)-\frac{1}{2}\ln(1-2\Upsilon),$$

where

$$\begin{gathered}\hat{E}_\upsilon(\upsilon,\xi)=\tfrac{1}{2}\hat{E}_{\xi\xi}(\upsilon,\xi),\\ \hat{E}\left(\upsilon,\overline{\Pi}(\upsilon)\right)=\frac{2\pi}{\ln\left(\frac{(1-2\upsilon)}{(1-2\Upsilon)}\right)}+\frac{2(\overline{\Pi}(\upsilon)+\theta)}{\ln(1-2\Upsilon)}\equiv\overline{e}(\upsilon),\\ \hat{E}\left(\upsilon,\underline{\Pi}(\upsilon)\right)=\frac{2\pi}{\ln\left(\frac{(1-2\upsilon)}{(1-2\Upsilon)}\right)}+\frac{2(\underline{\Pi}(\upsilon)+\theta)}{\ln(1-2\Upsilon)}\equiv\underline{e}(\upsilon),\\ \hat{E}(0,\xi)=0,\end{gathered}$$

$$\begin{gathered}\hat{F}_\upsilon(\upsilon,\xi)=\tfrac{1}{2}\hat{F}_{\xi\xi}(\upsilon,\xi),\\ \hat{F}\left(\upsilon,\overline{\Pi}(\upsilon)\right)=\frac{4\pi^2}{\left(\ln\left(\frac{(1-2\upsilon)}{(1-2\Upsilon)}\right)\right)^2}-\frac{4\left(\upsilon+(\overline{\Pi}(\upsilon)+\theta)^2\right)}{(\ln(1-2\Upsilon))^2}\equiv\overline{f}(\upsilon),\\ \hat{F}\left(\upsilon,\underline{\Pi}(\upsilon)\right)=\frac{4\pi^2}{\left(\ln\left(\frac{(1-2\upsilon)}{(1-2\Upsilon)}\right)\right)^2}-\frac{4\left(\upsilon+(\underline{\Pi}(\upsilon)+\theta)^2\right)}{(\ln(1-2\Upsilon))^2}\equiv\underline{f}(\upsilon),\\ \hat{F}(0,\xi)=0,\end{gathered}$$

$$\begin{gathered}\hat{G}_\upsilon(\upsilon,\xi)=\tfrac{1}{2}\hat{G}_{\xi\xi}(\upsilon,\xi),\\ \hat{G}\left(\upsilon,\overline{\Pi}(\upsilon)\right)=\tfrac{1}{2}\ln(1-2\upsilon)\equiv\overline{g}(\upsilon),\\ \hat{G}\left(\upsilon,\underline{\Pi}(\upsilon)\right)=\tfrac{1}{2}\ln(1-2\upsilon)\equiv\underline{g}(\upsilon),\\ \hat{G}(0,\xi)=0,\end{gathered}$$

$$\mathrm{SR}=\frac{\hat{E}(\Upsilon,\varpi)-\frac{2(\varpi+\theta)}{\ln(1-2\Upsilon)}}{\sqrt{\hat{F}(\Upsilon,\varpi)-\left(\hat{E}(\Upsilon,\varpi)\right)^2+\frac{4\left(\Upsilon+\ln(1-2\Upsilon)(\varpi+\theta)\hat{E}(\Upsilon,\varpi)\right)}{(\ln(1-2\Upsilon))^2}}},\tag{5}$$

$$\mathrm{DUR}=\hat{G}(\Upsilon,\varpi)-\frac{1}{2}\ln(1-2\Upsilon).\tag{6}$$

After the above transformations are performed, the problem becomes solvable by the method of heat potentials.

5 Expected duration of the trade

The method of heat potentials boils down to solving a system of Volterra equations of the second kind. However, there are certain quantities of interest, which can be calculated directly. One such quantity is the expected value of duration of a trade, which terminates only when the spread hits one of the barriers, $T=\infty$ (or $\Upsilon=0.5$). In this Section we show how to calculate this quantity analytically by solving an inhomogeneous linear ordinary differential equation (ODE).

In the case in question, the second change of variables is not necessary, so that we can concentrate on the following problem:

$$\begin{gathered}G_t(t,x)+(\theta-x)G_x(t,x)+\tfrac{1}{2}G_{xx}(t,x)=0,\\ G(t,\overline{\pi})=t,\quad G(t,\underline{\pi})=t,\end{gathered}$$

with an implicit terminal condition at $T\to\infty$. We can represent the corresponding solution in a semi-stationary form

$$G(t,x)=t+g(x),$$

where

$$(\theta - x)\, g_x(x) + \frac{1}{2} g_{xx}(x) = -1, \tag{7}$$

$$g(\overline{\pi}) = 0, \qquad g(\underline{\pi}) = 0. \tag{8}$$

We introduce $h(x) = g_x(x)$ and reduce the inhomogeneous second-order ordinary differential equation (7) to the inhomogeneous first order differential equation of the form

$$(\theta - x)\, h(x) + \frac{1}{2} h_x(x) = -1. \tag{9}$$

The latter equation (9) can be solved by the method of variation of constants:

$$h(x) = \left(c - 2\sqrt{\pi} N\left(\sqrt{2}(x-\theta)\right)\right) e^{(x-\theta)^2},$$

where c is an arbitrary constant. Taking into account boundary conditions (8), we can represent g as follows:

$$\begin{aligned} g(x) \quad = \quad & 2\Big(c e^{(x-\theta)^2} D(x-\theta) - \sqrt{\pi} N\left(\sqrt{2}(x-\theta)\right) e^{(x-\theta)^2} D(x-\theta) + \Delta(x-\theta) \\ & - c e^{(\underline{\pi}-\theta)^2} D(\underline{\pi}-\theta) + \sqrt{\pi} N\left(\sqrt{2}(\underline{\pi}-\theta)\right) e^{(\underline{\pi}-\theta)^2} D(\underline{\pi}-\theta) - \Delta(\underline{\pi}-\theta)\Big). \end{aligned}$$

where

$$c = \frac{\sqrt{\pi} N\left(\sqrt{2}(\overline{\pi}-\theta)\right) e^{(\overline{\pi}-\theta)^2} D(\overline{\pi}-\theta) - \Delta(\overline{\pi}-\theta) - \sqrt{\pi} N\left(\sqrt{2}(\underline{\pi}-\theta)\right) e^{(\underline{\pi}-\theta)^2} D(\underline{\pi}-\theta) + \Delta(\underline{\pi}-\theta)}{e^{(\overline{\pi}-\theta)^2} D(\overline{\pi}-\theta) - e^{(\underline{\pi}-\theta)^2} D(\underline{\pi}-\theta)}.$$

Here $D(x)$ is Dawson's function, and $\Delta(x)$ is its integral:

$$D(x) = e^{-x^2} \int_0^x e^{y^2} dy, \qquad \Delta(x) = \int_0^x D(y)\, dy.$$

The expected duration is given by the following expression:

$$\mathrm{DUR} = g(0). \tag{10}$$

We show the expected duration as a function of $\underline{\pi}, \overline{\pi}$ for $\theta = 1$ in Figure 1:

Figure 1 near here.

Given the fact that $\Upsilon \to 0.5$ corresponds to $T \to \infty$, we can see from this Figure that for sufficiently remote $\underline{\pi}, \overline{\pi}$ the process stays within the range $[\underline{\pi}, \overline{\pi}]$ indefinitely, or, at least, for a very long time.

The explicit expression for the expected duration given by Eq. (10) is interesting in its own right and also can be used for benchmarking solutions obtained via the method of heat potentials.

6 The method of heat potentials

Now we are ready to use the classical method of heat potentials to calculate the SR. Consider $\hat{E}$. We have to solve the following coupled system of Volterra integral equations:

$$\begin{aligned} & \underline{\varepsilon}(\upsilon) + \frac{1}{\sqrt{2\pi}} \int_0^{\upsilon} \frac{\left(\underline{\Pi}(\upsilon) - \underline{\Pi}(\zeta)\right) e^{-\frac{(\underline{\Pi}(\upsilon)-\underline{\Pi}(\zeta))^2}{2(\upsilon-\zeta)}}}{(\upsilon-\zeta)^{3/2}} \underline{\varepsilon}(\zeta)\, d\zeta \\ & + \frac{1}{\sqrt{2\pi}} \int_0^{\upsilon} \frac{\left(\underline{\Pi}(\upsilon) - \overline{\Pi}(\zeta)\right) e^{-\frac{(\underline{\Pi}(\upsilon)-\overline{\Pi}(\zeta))^2}{2(\upsilon-\zeta)}}}{(\upsilon-\zeta)^{3/2}} \overline{\varepsilon}(\zeta)\, d\zeta \quad = \quad \underline{e}(\upsilon), \end{aligned} \tag{11}$$

$$-\overline{\varepsilon}(\upsilon)+\frac{1}{\sqrt{2\pi}}\int_{0}^{\upsilon}\frac{\left(\overline{\Pi}(\upsilon)-\underline{\Pi}(\zeta)\right)e^{-\frac{\left(\overline{\Pi}(\upsilon)-\underline{\Pi}(\zeta)\right)^{2}}{2(\upsilon-\zeta)}}}{(\upsilon-\zeta)^{3/2}}\underline{\varepsilon}(\zeta)\,d\zeta \tag{12}$$

$$+\frac{1}{\sqrt{2\pi}}\int_{0}^{\upsilon}\frac{\left(\overline{\Pi}(\upsilon)-\overline{\Pi}(\zeta)\right)e^{-\frac{\left(\overline{\Pi}(\upsilon)-\overline{\Pi}(\zeta)\right)^{2}}{2(\upsilon-\zeta)}}}{(\upsilon-\zeta)^{3/2}}\overline{\varepsilon}(\zeta)\,d\zeta = \overline{e}(\upsilon),$$

Once these equations are solved, $\hat{E}(\upsilon,\xi)$ can be written as follows:

$$\hat{E}(\upsilon,\xi)=\frac{1}{\sqrt{2\pi}}\int_{0}^{\upsilon}\frac{(\xi-\underline{\Pi}(\zeta))\,e^{-\frac{(\xi-\underline{\Pi}(\zeta))^{2}}{2(\upsilon-\zeta)}}}{(\upsilon-\zeta)^{3/2}}\underline{\varepsilon}(\zeta)\,d\zeta+\frac{1}{\sqrt{2\pi}}\int_{0}^{\upsilon}\frac{\left(\xi-\overline{\Pi}(\zeta)\right)e^{-\frac{(\xi-\overline{\Pi}(\zeta))^{2}}{2(\upsilon-\zeta)}}}{(\upsilon-\zeta)^{3/2}}\overline{\varepsilon}(\zeta)\,d\zeta. \tag{13}$$

We can find $\hat{F}(\upsilon,\xi)$ by the same token:

$$\underline{\phi}(\upsilon)+\frac{1}{\sqrt{2\pi}}\int_{0}^{\upsilon}\frac{(\underline{\Pi}(\upsilon)-\underline{\Pi}(\zeta))\,e^{-\frac{(\underline{\Pi}(\upsilon)-\underline{\Pi}(\zeta))^{2}}{2(\upsilon-\zeta)}}}{(\upsilon-\zeta)^{3/2}}\underline{\phi}(\zeta)\,d\zeta \tag{14}$$

$$+\frac{1}{\sqrt{2\pi}}\int_{0}^{\upsilon}\frac{\left(\underline{\Pi}(\upsilon)-\overline{\Pi}(\zeta)\right)e^{-\frac{(\underline{\Pi}(\upsilon)-\overline{\Pi}(\zeta))^{2}}{2(\upsilon-\zeta)}}}{(\upsilon-\zeta)^{3/2}}\overline{\phi}(\zeta)\,d\zeta = \underline{f}(\upsilon),$$

$$-\overline{\phi}(\upsilon)+\frac{1}{\sqrt{2\pi}}\int_{0}^{\upsilon}\frac{\left(\overline{\Pi}(\upsilon)-\underline{\Pi}(\zeta)\right)e^{-\frac{(\overline{\Pi}(\upsilon)-\underline{\Pi}(\zeta))^{2}}{2(\upsilon-\zeta)}}}{(\upsilon-\zeta)^{3/2}}\underline{\phi}(\zeta)\,d\zeta \tag{15}$$

$$+\frac{1}{\sqrt{2\pi}}\int_{0}^{\upsilon}\frac{\left(\overline{\Pi}(\upsilon)-\overline{\Pi}(\zeta)\right)e^{-\frac{(\overline{\Pi}(\upsilon)-\overline{\Pi}(\zeta))^{2}}{2(\upsilon-\zeta)}}}{(\upsilon-\zeta)^{3/2}}\overline{\phi}(\zeta)\,d\zeta = \overline{f}(\upsilon),$$

$$\hat{F}(\upsilon,\xi)=\frac{1}{\sqrt{2\pi}}\int_{0}^{\upsilon}\frac{(\xi-\underline{\Pi}(\zeta))\,e^{-\frac{(\xi-\underline{\Pi}(\zeta))^{2}}{2(\upsilon-\zeta)}}}{(\upsilon-\zeta)^{3/2}}\underline{\phi}(\zeta)\,d\zeta+\frac{1}{\sqrt{2\pi}}\int_{0}^{\upsilon}\frac{\left(\xi-\overline{\Pi}(\zeta)\right)e^{-\frac{(\xi-\overline{\Pi}(\zeta))^{2}}{2(\upsilon-\zeta)}}}{(\upsilon-\zeta)^{3/2}}\overline{\phi}(\zeta)\,d\zeta. \tag{16}$$

In particular,

$$\begin{aligned}\hat{E}(\Upsilon,\varpi) &= \frac{1}{\sqrt{2\pi}}\int_{0}^{\Upsilon}\frac{(\varpi-\underline{\Pi}(\zeta))\,e^{-\frac{(\varpi-\underline{\Pi}(\zeta))^{2}}{2(\Upsilon-\zeta)}}}{(\Upsilon-\zeta)^{3/2}}\underline{\varepsilon}(\zeta)\,d\zeta+\frac{1}{\sqrt{2\pi}}\int_{0}^{\Upsilon}\frac{\left(\varpi-\overline{\Pi}(\zeta)\right)e^{-\frac{(\varpi-\overline{\Pi}(\zeta))^{2}}{2(\Upsilon-\zeta)}}}{(\Upsilon-\zeta)^{3/2}}\overline{\varepsilon}(\zeta)\,d\zeta,\\ \hat{F}(\Upsilon,\varpi) &= \frac{1}{\sqrt{2\pi}}\int_{0}^{\Upsilon}\frac{(\varpi-\underline{\Pi}(\zeta))\,e^{-\frac{(\varpi-\underline{\Pi}(\zeta))^{2}}{2(\Upsilon-\zeta)}}}{(\Upsilon-\zeta)^{3/2}}\underline{\phi}(\zeta)\,d\zeta+\frac{1}{\sqrt{2\pi}}\int_{0}^{\Upsilon}\frac{\left(\varpi-\overline{\Pi}(\zeta)\right)e^{-\frac{(\varpi-\overline{\Pi}(\zeta))^{2}}{2(\Upsilon-\zeta)}}}{(\Upsilon-\zeta)^{3/2}}\overline{\phi}(\zeta)\,d\zeta.\end{aligned}$$

It is important to notice that $(\underline{\varepsilon}(\zeta),\overline{\varepsilon}(\zeta))$ and $\left(\underline{\phi}(\zeta),\overline{\phi}(\zeta)\right)$ are singular at $\zeta=\Upsilon$. However, due to the dampening impact of the exponents $\exp\left(-\left(\varpi-\overline{\Pi}(\zeta)\right)^{2}/2(\Upsilon-\zeta)\right)$, the corresponding integrals still converge.

We now know $\hat{E}(\Upsilon,\varpi),\hat{F}(\Upsilon,\varpi)$ and calculate the SR by using Eq. (5). $\hat{G}(\Upsilon,\varpi)$ and DUR can be calculated in a similar fashion.

7 Numerical method

To compute the SR, we need to find $\hat{E}(\Upsilon,\varpi)$ and $\hat{F}(\Upsilon,\varpi)$, and then apply Eq. (5). $\hat{E}(\Upsilon,\varpi)$ and $\hat{F}(\Upsilon,\varpi)$ can be computed using Eqs (13) and (16) by simple integration with pre-computed $(\underline{\varepsilon},\overline{\varepsilon})$ and $\left(\underline{\phi},\overline{\phi}\right)$. In this section, we develop a numerical method to compute these quantities by solving Eqs (11)–(12), and (14)–(15) by extending the methods described in Lipton and Kaushansky, [6, 8]. For illustrative purposes we develop a simple scheme based on the trapezoidal rule for Stieltjes integrals.

We want to solve a generic system of the form:

$$\begin{aligned} \nu^1(\upsilon) + \int_0^\upsilon \frac{K^{1,1}(\upsilon,s)}{\sqrt{\upsilon-s}} \nu^1(s)\,ds + \int_0^\upsilon K^{1,2}(\upsilon,s)\nu^2(s)\,ds &= \chi^1(\upsilon), \\ -\nu^2(\upsilon) + \int_0^\upsilon K^{2,1}(\upsilon,s)\nu^1(s)\,ds + \int_0^\upsilon \frac{K^{2,2}(\upsilon,s)}{\sqrt{\upsilon-s}} \nu^2(s)\,ds &= \chi^2(\upsilon), \end{aligned} \tag{17}$$

with respect to variables $(\nu^1(\upsilon), \nu^2(\upsilon))$, where

$$\begin{aligned} K^{1,1}(\upsilon,s) &= \frac{1}{\sqrt{2\pi}} \frac{\underline{\Pi}(\upsilon)-\underline{\Pi}(s)}{\upsilon-s} \exp\left(-\frac{(\underline{\Pi}(\upsilon)-\underline{\Pi}(s))^2}{2(\upsilon-s)}\right), \\ K^{1,2}(\upsilon,s) &= \frac{1}{\sqrt{2\pi}} \frac{\underline{\Pi}(\upsilon)-\overline{\Pi}(s)}{(\upsilon-s)^{3/2}} \exp\left(-\frac{(\underline{\Pi}(\upsilon)-\overline{\Pi}(s))^2}{2(\upsilon-s)}\right), \\ K^{2,1}(\upsilon,s) &= \frac{1}{\sqrt{2\pi}} \frac{\overline{\Pi}(\upsilon)-\underline{\Pi}(s)}{(\upsilon-s)^{3/2}} \exp\left(-\frac{(\overline{\Pi}(\upsilon)-\underline{\Pi}(s))^2}{2(\upsilon-s)}\right), \\ K^{2,2}(\upsilon,s) &= \frac{1}{\sqrt{2\pi}} \frac{\overline{\Pi}(\upsilon)-\overline{\Pi}(s)}{\upsilon-s} \exp\left(-\frac{(\overline{\Pi}(\upsilon)-\overline{\Pi}(s))^2}{2(\upsilon-s)}\right). \end{aligned} \tag{18}$$

It is clear that

$$\begin{aligned} K^{1,1}(\upsilon,\upsilon) &= \frac{1}{\sqrt{2\pi}} \lim_{s\to\upsilon} \frac{\underline{\Pi}(\upsilon)-\underline{\Pi}(s)}{\upsilon-s} = \frac{\theta-\underline{\pi}}{\sqrt{2\pi}\sqrt{1-2\upsilon}}, \\ K^{1,2}(\upsilon,\upsilon) &= 0, \\ K^{2,1}(\upsilon,\upsilon) &= 0, \\ K^{2,2}(\upsilon,\upsilon) &= \frac{1}{\sqrt{2\pi}} \lim_{s\to\upsilon} \frac{\overline{\Pi}(\upsilon)-\overline{\Pi}(s)}{\upsilon-s} = \frac{\theta-\overline{\pi}}{\sqrt{2\pi}\sqrt{1-2\upsilon}}. \end{aligned} \tag{19}$$

We can equally rewrite the relevant integrals as Stieltjes integrals

$$\begin{aligned} \nu^1(\upsilon) - 2\int_0^\upsilon K^{1,1}(\upsilon,s)\nu^1(s)\,d\sqrt{\upsilon-s} + \int_0^\upsilon K^{1,2}(\upsilon,s)\nu^2(s)\,ds &= \chi^1(\upsilon), \\ -\nu^2(\upsilon) + \int_0^\upsilon K^{2,1}(\upsilon,s)\nu^1(s)\,ds - 2\int_0^\upsilon K^{2,2}(\upsilon,s)\nu^2(s)\,d\sqrt{\upsilon-s} &= \chi^2(\upsilon). \end{aligned} \tag{20}$$

Consider a grid $0 = \upsilon_0 < \upsilon_1 < \ldots < \upsilon_n = \Upsilon$, and let $\Delta_{k,l} = \upsilon_k - \upsilon_l$. Then, using the trapezoidal rule for approximation of integrals, we get the following approximation of last two equations:

$$\begin{aligned} \nu_k^1 + \sum_{i=1}^k \left(\frac{\left(K_{k,i}^{1,1}\nu_i^1 + K_{k,i-1}^{1,1}\nu_{i-1}^1\right)}{\left(\sqrt{\Delta_{k,i}} + \sqrt{\Delta_{k,i-1}}\right)} + \frac{1}{2}\left(K_{k,i}^{1,2}\nu_i^2 + K_{k,i-1}^{1,2}\nu_{i-1}^2\right)\right)\Delta_{i,i-1} &= \chi_k^1, \\ -\nu_k^2 + \sum_{i=1}^k \left(\frac{1}{2}\left(K_{k,i}^{2,1}\nu_i^1 + K_{k,i-1}^{2,1}\nu_{i-1}^1\right) + \frac{\left(K_{k,i}^{2,2}\nu_i^2 + K_{k,i-1}^{2,2}\nu_{i-1}^2\right)}{\left(\sqrt{\Delta_{k,i}} + \sqrt{\Delta_{k,i-1}}\right)}\right)\Delta_{i,i-1} &= \chi_k^2. \end{aligned} \tag{21}$$

where

$$\nu_i^\alpha = \nu^\alpha(\upsilon_i), \qquad \chi_i^\alpha = \chi^\alpha(\upsilon_i), \qquad K_{k,j}^{\alpha,\beta} = K^{\alpha,\beta}(\upsilon_k,\upsilon_i), \qquad \alpha,\beta = 1,2. \tag{22}$$

Taking into account that

$$\begin{aligned} \left(\nu_0^1, \nu_0^2\right) &= \left(\chi_0^1, -\chi_0^2\right), \\ \left(\nu_1^1, \nu_1^2\right) &= \left(\frac{\chi_1^1}{\left(1 + K_{1,1}^{1,1}\sqrt{\upsilon_1}\right)}, -\frac{\chi_1^2}{\left(1 - K_{1,1}^{2,2}\sqrt{\upsilon_1}\right)} \right), \end{aligned} \tag{23}$$

and assuming that $\left(\nu_2^1,\nu_2^2\right),\ldots,\left(\nu_{k-1}^1,\nu_{k-1}^2\right)$ have been computed, we can easily find $\left(\nu_k^1,\nu_k^2\right)$:

$$\begin{aligned}\nu_k^1 &= \left(1+K_{k,k}^{1,1}\sqrt{\Delta_{k,k-1}}\right)^{-1}\left(\chi_k^1-K_{k,k-1}^{1,1}\nu_{k-1}^1\sqrt{\Delta_{k,k-1}}-\frac{1}{2}K_{k,k-1}^{1,2}\nu_{k-1}^2\Delta_{k,k-1}\right.\\ &\left.-\sum_{i=1}^{k-1}\left(\frac{\left(K_{k,i}^{1,1}\nu_i^1+K_{k,i-1}^{1,1}\nu_{i-1}^1\right)}{\left(\sqrt{\Delta_{k,i}}+\sqrt{\Delta_{k,i-1}}\right)}+\frac{1}{2}\left(K_{k,i}^{1,2}\nu_i^2+K_{k,i-1}^{1,2}\nu_{i-1}^2\right)\right)\Delta_{i,i-1}\right),\\ \nu_k^2 &= \left(-1+K_{k,k}^{2,2}\sqrt{\Delta_{k,k-1}}\right)^{-1}\left(\chi_k^2-\frac{1}{2}K_{k,k-1}^{2,1}\nu_{k-1}^1\Delta_{k,k-1}-K_{k,k-1}^{2,2}\nu_{k-1}^2\sqrt{\Delta_{k,k-1}}\right.\\ &\left.-\sum_{i=1}^{k-1}\left(\frac{1}{2}\left(K_{k,i}^{2,1}\nu_i^1+K_{k,i-1}^{2,1}\nu_{i-1}^1\right)+\frac{\left(K_{k,i}^{2,2}\nu_i^2+K_{k,i-1}^{2,2}\nu_{i-1}^2\right)}{\left(\sqrt{\Delta_{k,i}}+\sqrt{\Delta_{k,i-1}}\right)}\right)\Delta_{i,i-1}\right).\end{aligned} \tag{24}$$

The approximation error of the integrals is of order $O(\Delta^2)$, where $\Delta=\max_i\Delta_{i,i-1})$. Hence, on uniform grid, the convergence is of order $O(\Delta)$. We emphasize that, due to the nature of $(\underline{e}(\upsilon),\overline{e}(\upsilon))$, etc., it is necessary to use a highly inhomogeneous grid which is concentrated near the right endpoint.

7.1 Computation of the Sharpe ratio

Once $(\underline{\varepsilon}(\upsilon),\overline{\varepsilon}(\upsilon))$ and $\left(\underline{\phi}(\upsilon),\overline{\phi}(\upsilon)\right)$ are computed, we can approximate $\hat{E}(\upsilon,\xi)$ and $\hat{F}(\upsilon,\xi)$. We interested to compute these functions at one point (Υ,ϖ), which can be done by approximation of the integrals using the trapezoidal rule:

$$\hat{E}(\Upsilon,\varpi)=\frac{1}{2}\sum_{i=1}^{k}\left(\underline{w}_{n,i}\underline{\varepsilon}_i+\underline{w}_{n,i-1}\underline{\varepsilon}_{i-1}+\ \overline{w}_{n,i}\overline{\varepsilon}_i+\ \overline{w}_{n,i-1}\overline{\varepsilon}_{i-1}\right)\Delta_{i,i-1}, \tag{25}$$

and

$$\hat{F}(\Upsilon,\varpi)=\frac{1}{2}\sum_{i=1}^{k}\left(\underline{w}_{n,i}\underline{\phi}_i+\underline{w}_{n,i-1}\underline{\phi}_{i-1}+\ \overline{w}_{n,i}\overline{\phi}_i+\ \overline{w}_{n,i-1}\overline{\phi}_{i-1}\right)\Delta_{i,i-1}. \tag{26}$$

The corresponding weights are as follows:

$$\begin{aligned}\underline{w}_{n,i}&=\frac{(\varpi-\underline{\Pi}_i)e^{-\frac{(\varpi-\underline{\Pi}_i)^2}{2\Delta_{n,i}}}}{\sqrt{2\pi}\Delta_{n,i}^{3/2}}, &\overline{w}_{n,i}&=\frac{(\varpi-\overline{\Pi}_i)e^{-\frac{(\varpi-\overline{\Pi}_i)^2}{2\Delta_{n,i}}}}{\sqrt{2\pi}\Delta_{n,i}^{3/2}} & 1\le i<n,\\ \underline{w}_{n,i}&=0 & \overline{w}_{n,i}&=0, & i=n.\end{aligned}$$

As a result, we get the following algorithm for the numerical evaluation of the SR.

Algorithm 1 Numerical evaluation of the Sharpe ratio

Step 1	Define a time grid $0=\upsilon_0<\upsilon_1<\ldots<\Upsilon$.
Step 2	Compute $\underline{\epsilon}(\upsilon),\overline{\epsilon}(\upsilon),\underline{\phi}(\upsilon),\overline{\phi}(\upsilon)$ using numerical method in Section 7.
Step 3	Compute $\hat{E}(\Upsilon,\varpi)$ using (25).
Step 4	Compute $\hat{F}(\Upsilon,\varpi)$ by using Eq. (26).
Step 5	Compute the Sharpe ratio by using Eq. (5).

8 Numerical results

8.1 Comparison with Monte Carlo simulations

We compute the SR for various values of $\underline{\pi}$ and $\overline{\pi}$, and as a result show the SR as a function of $(\underline{\pi},\overline{\pi})$. After that one can choose $(\underline{\pi},\overline{\pi})$ in order to maximize the SR.

To be concrete, consider $\theta = 1.0$ and $\Upsilon = 0.49$, $T = 1.96$. We compare our results with Monte Carlo method, which simulates the process and compute its expectation and variance (see [12]). First, we compare separately E, $\sigma = \sqrt{F - E^2}$, and G calculated by both methods in Figure 2:

Figure 2 near here.

Second, we show the results for the SR itself in Figure 3:

Figure 3 near here.

We see that the relative difference between the method of heat potentials and the Monte Carlo method is small and mainly comes from the Monte Carlo noise.

8.2 Optimization of the Sharpe ratio

In this section we solve a problem of finding parameters to maximize the SR by analyzing it as a function of $(\underline{\pi}, \overline{\pi})$ for different values of θ and Υ. Two problems are considered: (A) Fix Υ and maximize the SR over $(\underline{\pi}, \overline{\pi})$; (B) Maximize the SR over $(\underline{\pi}, \overline{\pi}, \Upsilon)$.

Given that the natural unit $\Omega = 1/\sqrt{2}$, we consider three representative values of θ, namely $\theta = 1$, $\theta = 0.5$, and $\theta = 0$, corresponding to strong and weak mispricing and fair pricing, respectively. We choose three maturities, $\Upsilon = 0.49, 0.4999, 0.499999$ or ,equivalently, $T = 1.96, 4.26, 6.56$. For negative θ, the corresponding SR can be obtained by reflection if needed:

Figure 4 near here.

Figure 5 near here.

Figure 6 near here.

The optimal bounds $(\underline{\pi}^*, \overline{\pi}^*)$ are given in the Table 1 below:

Table 1 near here.

This table shows that in the case when the original mispricing is strong ($\theta = 1$) it is not optimal to stop the trade early. When the mispricing is weaker ($\theta = 0.5$) or there is no mispricing in the first place ($\theta = 0$) it is not optimal to stop losses, but it might be beneficial to take profits. We emphasize that in practice one needs to use a highly reliable estimation of the O-U parameters to be able to use these rules with confidence.

9 Conclusions

In this paper we create an analytical framework for computing optimal stop-loss/take-profit bounds $(\underline{\pi}^*, \overline{\pi}^*)$ for O-U driven trading strategies by using the method of heat potentials.

First, we present a method for calibrating the corresponding O-U process to market prices. Second, we derive an explicit expression for the SR given by Eq. (5), and maximize it with respect to the stop loss/ take profit bounds $(\underline{\pi}, \overline{\pi})$. Third, for three representative values of θ we calculate the SR on a grid of $(\underline{\pi}, \overline{\pi})$ and pre-chosen times and graphically summarize in Figures 4, 5, 6. Next, for each case we perform optimization and present $(\underline{\pi}^*, \overline{\pi}^*)$ in Table 1. In agreement with intuition, in the case of strong misprising it is optimal to wait until the trade's expiration without imposing stop losses/ take profit bounds. For weaker mispricing, it is not optimal to stop losses, but it might be optimal to take profits early.

Our rules help liquidity providers to decide how to offer liquidity to the market in the most profitable way, as well as by statistical arbitrage traders to optimally execute their trading strategies.

A very interesting and difficult multi-dimensional version of these rules (covering several correlated stocks) will be described elsewhere.

Acknowledgement 1 *We greatly appreciate valuable discussions with Dr. Marsha Lipton, our partner at Investimizer.*

Acknowledgement 2 *We are grateful to Dr. Vadim Kaushansky for his help with an earlier version of this paper.*

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$\theta \setminus \Upsilon, T$	$0.49,\ 0.8$	$0.4999,\ 1.96$	$0.499999,\ 4.26$
1.0	$\underline{\pi}^* = -4.0$ $\overline{\pi}^* = 4.0$ SR = 1.2261	$\underline{\pi}^* = -4.0$ $\overline{\pi}^* = 4.0$ SR = 1.3824	$\underline{\pi}^* = -4.0$ $\overline{\pi}^* = 4.0$ SR = 1.3709
0.5	$\underline{\pi}^* = -4.0$ $\overline{\pi}^* = 0.6$ SR = 0.8219	$\underline{\pi}^* = -4.0$ $\overline{\pi}^* = 0.9$ SR = 0.8792	$\underline{\pi}^* = -4.0$ $\overline{\pi}^* = 1.0$ SR = 0.8963
0.0	$\underline{\pi}^* = -4.0$ $\overline{\pi}^* = 0.1$ SR = 0.7075	$\underline{\pi}^* = -4.0$ $\overline{\pi}^* = 0.4$ SR = 0.7139	$\underline{\pi}^* = -4.0$ $\overline{\pi}^* = 0.1$ SR = 0.7411

Table 1: The Sharpe Ratio maximized over $(\underline{\pi}, \overline{\pi})$ for fixed Υ or T.

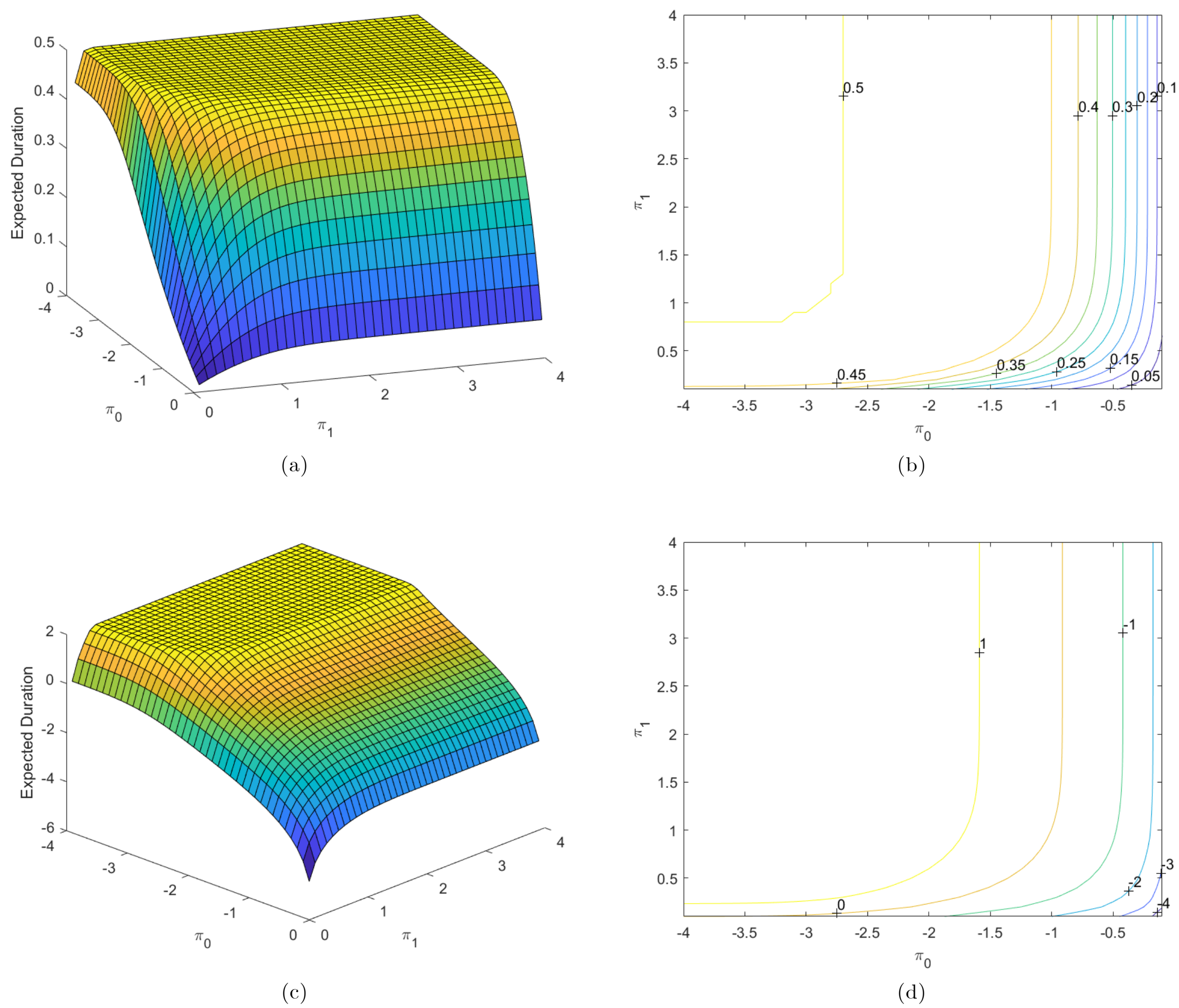
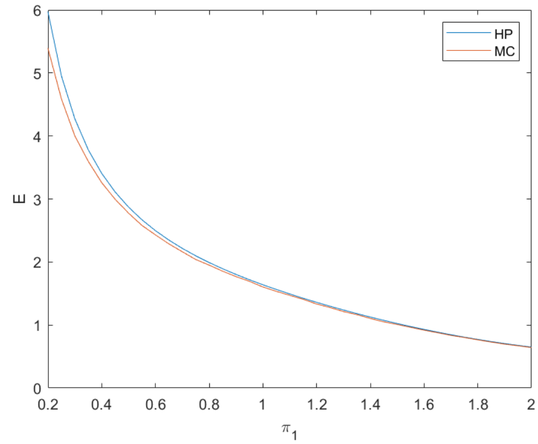
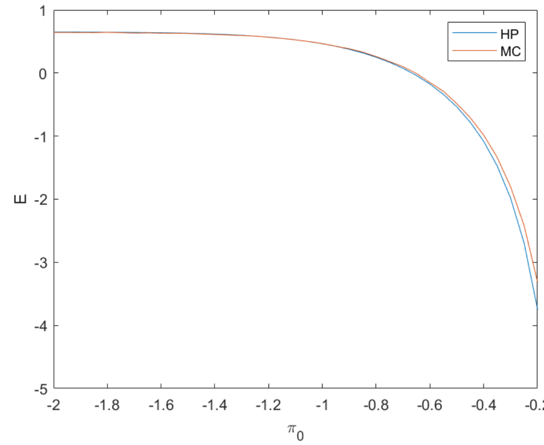


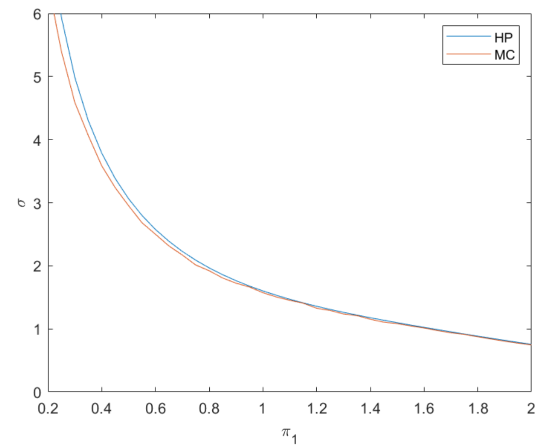
Figure 1: In Figures (a)-(b) we show the expected duration $\Upsilon = (1 - \exp(-2G)/2)$ as a function of $\underline{\pi}$, $\overline{\pi}$; in Figures (c)-(d) we show the logarithm of the expected duration G. The corresponding $\theta = 1.0$. Here and in Figures 4, 5, 6 $\pi_0 \equiv \underline{\pi}$, $\pi_1 \equiv \overline{\pi}$.



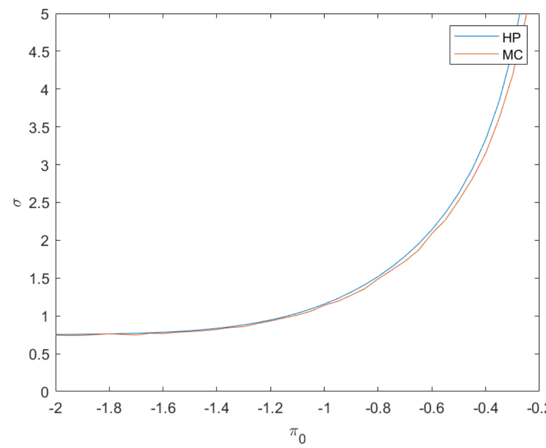
(a)



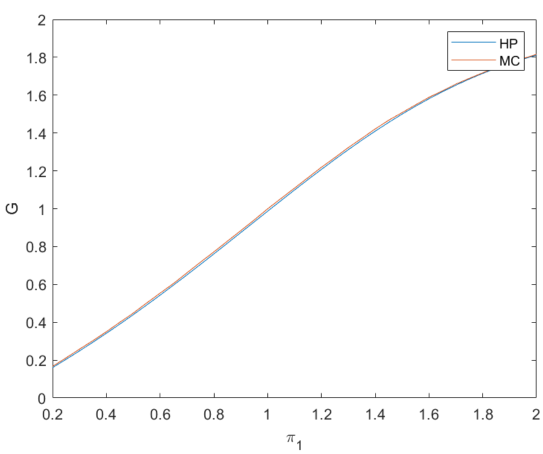
(b)



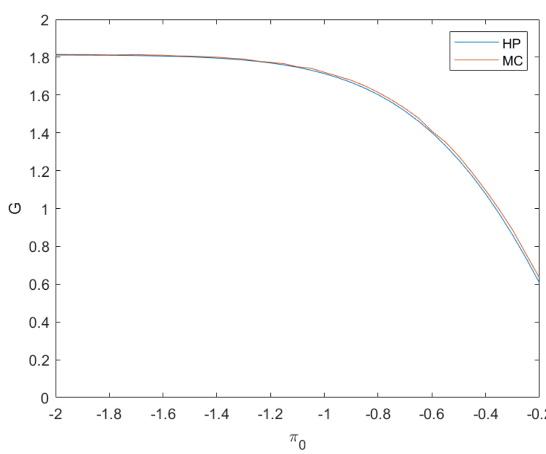
(c)



(d)

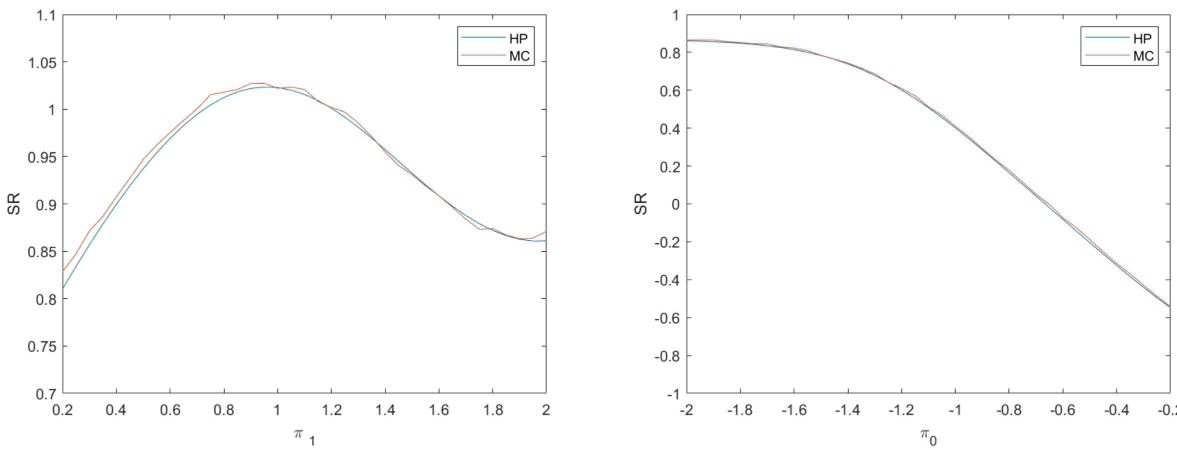


(e)



(f)

Figure 2: (a) $E, \sigma = \sqrt{F - E^2}$ and G as functions of $\overline{\pi} \equiv \pi_1$ computed by using the method of heat potentials and the Monte Carlo method for $\underline{\pi} = -2$; (b) Same quantities as functions of $\underline{\pi} \equiv \pi_0$ computed using the method of heat potentials and the Monte Carlo method for $\overline{\pi} = 1$; $\theta = 1.0$, $T = 1.96$.



(a) (b)

Figure 3: (a) The Sharpe ratio as a function of $\overline{\pi} \equiv \pi_1$ computed by using the method of heat potentials and the Monte Carlo method for $\underline{\pi} = -1$ (b) the Sharpe ratio as a function of $\underline{\pi} \equiv \pi_0$ computed using the method of heat potentials and the Monte Carlo method for $\overline{\pi} = 1$; $\theta = 1.0$, $T = 1.96$.

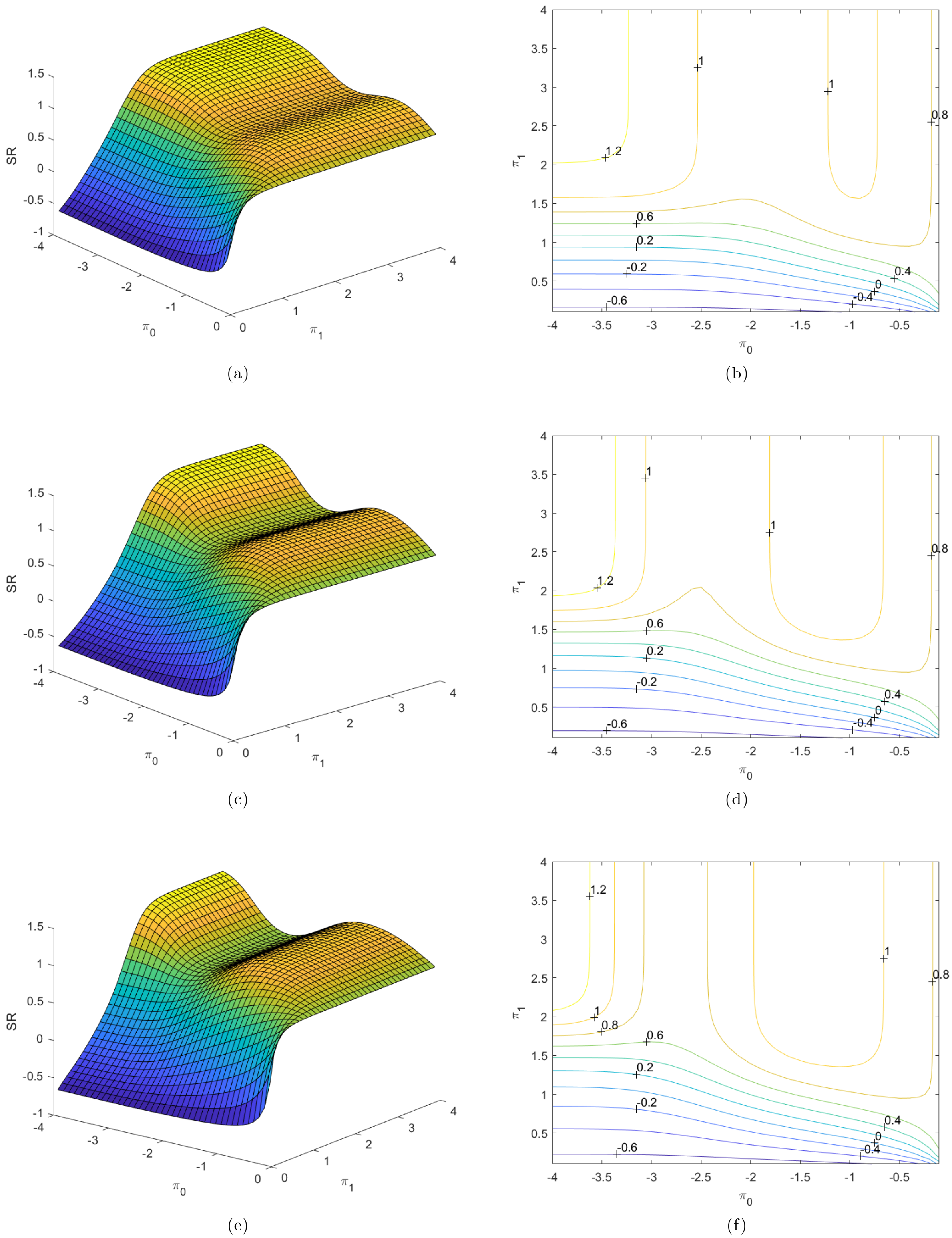


Figure 4: The Sharpe Ratio as a function of $(\underline{\pi}, \overline{\pi})$ for $\theta = 1.0$ a)-b) $T = 1.96$, c)-d) $T = 4.26$, e)-f) $T = 6.56$.

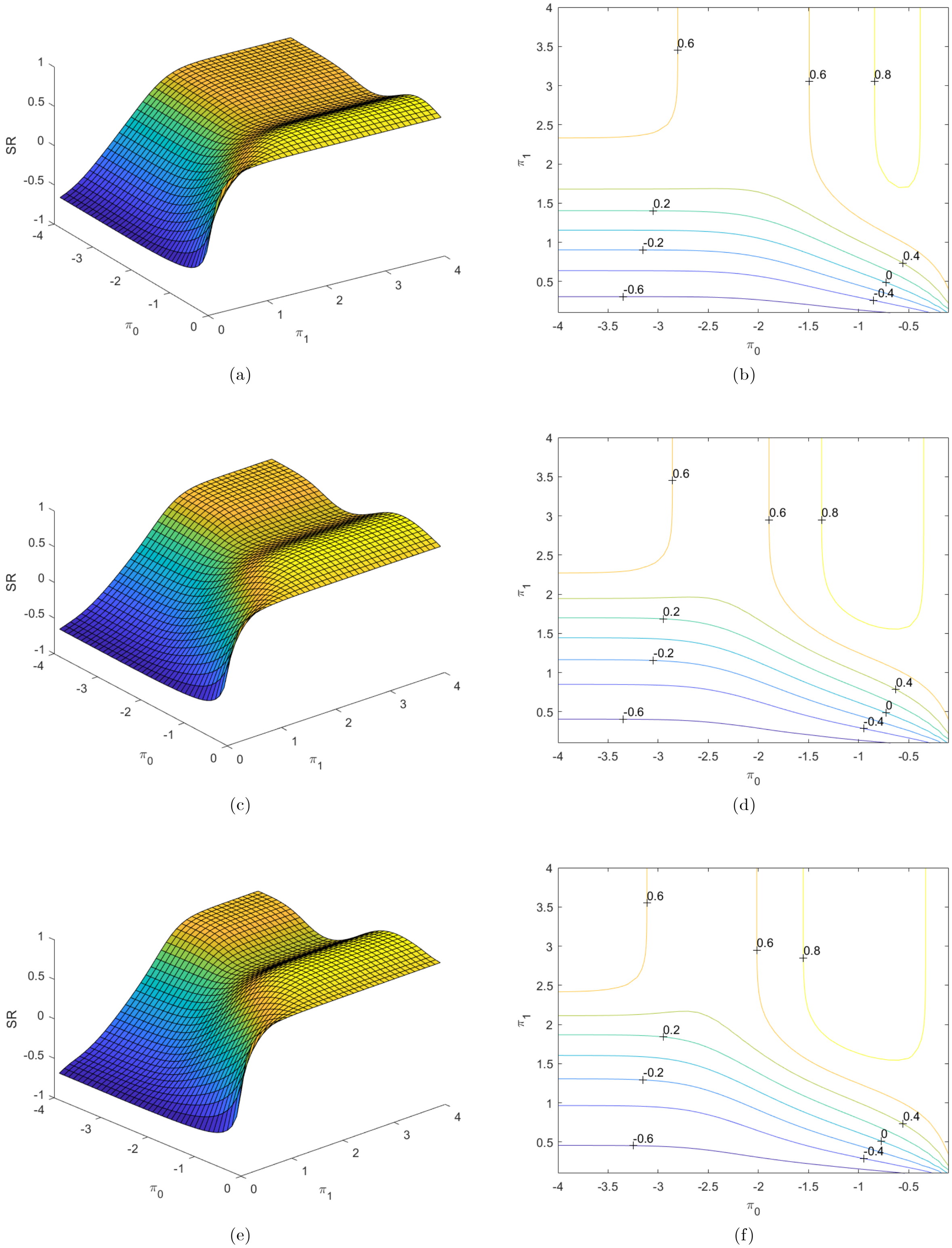


Figure 5: The Sharpe Ratio as a function of $(\underline{\pi}, \overline{\pi})$ for $\theta = 0.5$ a)-b) $T = 1.96$, c)-d) $T = 4.26$, e)-f) $T = 6.56$.

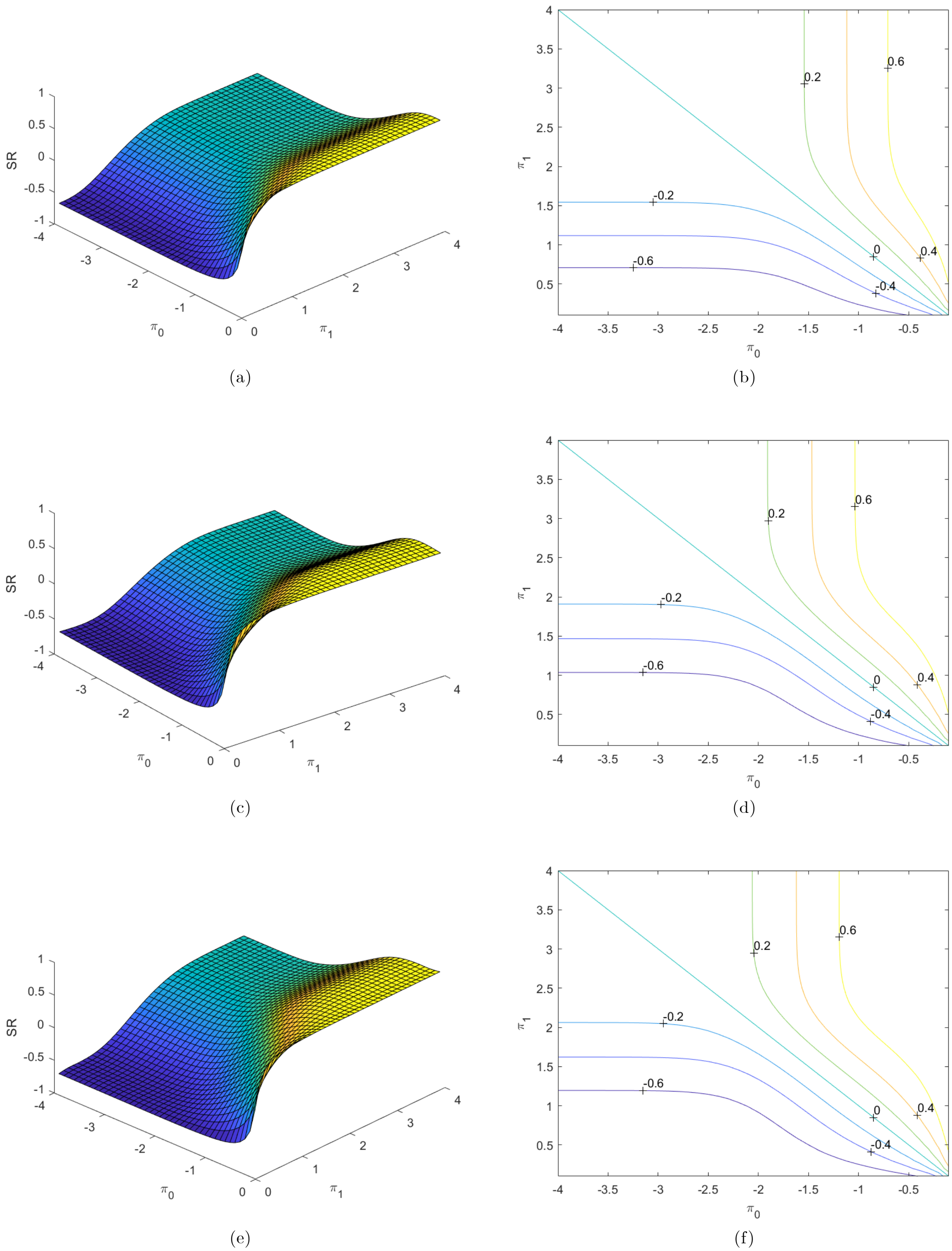


Figure 6: The Sharpe Ratio as a function of $(\underline{\pi}, \overline{\pi})$ for $\theta = 0.0$ a)-b) $T = 1.96$, c)-d) $T = 4.26$, e)-f) $T = 6.56$.